Analysis of symmetry breaking in quartz blocks using superstatistical random matrix theory

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(Date textdate; Received textdate; Revised textdate; Accepted textdate; Published textdate)

Abstract

We study the symmetry breaking of acoustic resonances measured by Ellegaard et al. [1] in quartz blocks. The observed resonance spectra show a gradual transition from a superposition of two uncoupled components, one for each symmetry realization, to a single component well represented by a Gaussian orthogonal ensemble (GOE) of random matrices. We discuss the applicability of superstatistical random-matrix theory to the final stages of the symmetry breaking transition. A comparison is made between different formula of the superstatistics and a pervious work [8], which describes the same data by introducing a third GOE component. Our results suggest that the inverse-chi-square superstatistics could be used for studying the whole symmetry breaking process.

INTRODUCTION

Few years ago Ellegaard et al. studied the influence of the flip symmetry breaking in quartz blocks on their spectral statistics [1]. Crystalline quartz exhibits a D_3 point-group symmetry about the crystal's Z (optical) axis and three two-fold rotation symmetries about the three X (piezoelectric) axes; the latter three axes lie in a plane orthogonal to the Z axis and sustain angles of 120° with respect to each other. Ellegaard et al.[1] used rectangular blocks of dimensions $14 \times 25 \times 40$ mm^3, cut in such a way that all symmetries are fully broken except a two-fold 'flip' symmetry about one of the three X axes. In order to break the flip symmetry, Ellegaard et al removed an octant of a sphere with a successfully increasing radius from one corner of the rectangular block. The breaking of the flip symmetry causes the eigenmodes belonging to the two different representations to interact and mix. As the radius is made larger, the nearest-neighbor-spacing (NNS) distribution makes a fast transition from a distribution of two non-interacting spectra towards the distribution of one chaotic system without symmetries. Full chaos was attained at an octant radius of $r = 10$ mm, as the cut block had the shape of a three -dimensional Sinai Billiard.

A problem like this is found, e.g., in nuclear physics where isospin symmetry, characteristic of the strong interactions, is only approximate due to Coulomb effects [3]. Isospin mixing was analyzed by Guhr and Weidenmüller using a random matrix approach [4]. They used a random matrix model to describe experimental data and to estimate the average symmetry-breaking matrix element, i.e., the average Coulomb matrix element. The random matrix ensemble consists of matrices $H = H_0 + V$, where H_0 is block-diagonal, here two blocks for the two-fold symmetry, each a member of the GOE; V couples the blocks, breaking the symmetry. Level statistics were studied within this model analytically by Leitner [5]. He assumed that the probability density of spacing belonging to the same block are unaffected by the coupling matrix V. He took into account the interaction between pairs of levels belonging to different blocks, yielding linear dependence of the spacing distribution near the origin. Taking into account the interaction between more than two levels might lead to fractional-level repulsion as recently shown by Bäcker et al. [6]. While Leitner's formalism is a perturbation-theory approach, which is valid only for small V, he has used it as a model for the whole symmetry breaking process in quartz-block data [7]. El-Hady et al. [8] described the transition by introducing a third GOE sequence which increased in the expense of the

initial two independent sequences. To fit the data in Ref. [1], the authors assumed that the

initial sequence are pseudointegrable (for definition, see the discussion following Eq. (12) below). This independent-sequence model of symmetry breaking was further elaborated in [9], where a relation between the fractional level density of each sequence is related to the mean squared symmetry-breaking matrix element. De Carvalho et al. [10] take into account the interaction of the three level sequences using the perturbation method suggested by Leitner [5].

In the present paper we perform an analysis of data in Ref. [1] using a superstatistical generalization of Random Matrix Theory (RMT) [2, 11–14]. We are motivated by the success of the superstatistical approach to describe the final stages of the transition from integrability to chaos. A brief account of this approach is given in Section 2. Section 3 reports the results of the superstatistical analysis of symmetry breaking. We also compare the results our work with that of Ref. [8]. A discussion of the results of this contribution is given in Section 4.

SUPERSTATISTICAL RMT

The concept of superstatistics was introduced by Beck and Cohen [15] to describe deviation of thermodynamic system from equilibrium. This concept has been successfully applied to a wide variety of physical problems, including turbulence [16], plasma physics [17], cosmic-ray statistics [18], cancer survival [19], and econophysics [20]. The application of the superstatistics to RMT [21] assumes the spectrum of a mixed system as made up of many smaller cells that are temporarily in a chaotic phase. Each cell is large enough to obey the statistical requirements of RMT but has a different distribution parameter η associated with it, according to a probability density $f(\eta)$. Consequently, the superstatistical random matrix ensemble that describes the mixed system is a mixture of Gaussian ensembles. The joint probability density distribution of the matrix elements is obtained by integrating distributions of the form

$$P(H) = \frac{1}{Z(\eta)} exp[-\eta Tr(H^{\dagger} H)] \tag{1}$$

over all positive values with a statistical weight $f(\eta)$, which leads to

$$P(H) = \int_0^\infty f(\eta) \frac{exp[-\eta Tr(H^\dagger H)]}{Z(\eta)} d\eta \tag{2}$$

where $Z(\eta) = \int exp[-\eta Tr(H^\dagger H)] dH$,

One of the fundamental assumptions of RMT is that the matrix element joint probability density distribution is base independent. This makes the theory suitable for modelling quantum chaotic systems. Indeed, the eigenfunctions of a Hamiltonian of a system with a chaotic classical limit are unknown in principle. In other words, there is no special basis to express the eigenstates of a chaotic system. In integrable systems, on the other hand, the eigenbasis of the Hamiltonian is known in principle. In this basis, each eigenfunction has just one component that obviously indicates the absence of complexity. In the nearly ordered regime, mixing of quantum states belonging to adjacent levels can be ignored and the energy levels are uncorrelated.

Parameter distribution

The distribution $f(\eta)$ is determined by the spatiotemporal dynamics of the entire system under consideration. Beck et al.[22] have argued that typical experimental data are described by one of three superstatistical universality classes, namely, χ^2, inverse χ^2 or log-normal superstatistics. The first universality holds if η has contribution from ν Gaussian random variables X_1, ... , X_υ due to various relevant degrees of freedom in the system. Then a positive η is obtained by setting $\eta = \sum_{i=1}^{\upsilon} X_i^2$ is and $f(\eta/\eta_0)$ is a χ^2 distributed with degree ν,

$$f(\eta/\eta_0) = \frac{1}{\Gamma(\nu/2)} (\frac{\nu}{2\eta_0})^{\nu/2} \eta^{\nu/2-1} e^{-\nu\eta/2\eta_0} \tag{3}$$

Here, η_0 is the average value of η, with $\eta_0 = \int_0^\infty \eta f(\eta/\eta_0) d\eta$

In the second universality class, the same considerations as above can be applied if η^{-1}, rather than η, is the sum of several squared Gaussian random variables. The resulting $f(\eta/\eta_0)$ is the inverse- χ^2 distribution given by

$$f(\eta/\eta_0) = \frac{\eta_0}{\Gamma(\nu/2)} (\frac{\nu\eta_0}{2})^{\nu/2} \eta^{-\nu/2-2} e^{-\nu\eta_0/2\eta} \tag{4}$$

In the third universality class, instead of being a sum of many contributions, the random variable η may be generated by multiplicative random processes, i.e. $\eta = \prod_{i=1}^{\nu} X_i$. Then

$\ln \eta = \sum_{i=1}^{\nu} \ln X_i$ is a sum of Gaussian random variables; hence it is Gaussian as well. Thus is log-normally distributed, i.e.,

$$f(\eta/\eta_0) = \frac{1}{\sqrt{2\pi}\nu\eta} e^{-[\ln(\frac{\eta}{\mu})]^2/2\nu^2}, \tag{5}$$

which has an average $\mu\sqrt{\psi}$ and variance $\sigma^2 = \mu^2\psi(\psi - 1)$, where $\psi = \exp(\nu^2)$.

NNS distribution

It follows from Eq.(2) that the statistical measures of the eigenvalues of the superstatistical ensemble are obtained as an average of the corresponding η-dependent ones of standard RMT weighted with the parameter distribution $f(\eta/\eta_0)$. In particular, the superstatistical NNS distribution is given by [21] as

$$p(s) = \int_0^\infty f(\eta/\eta_0) p_w(\eta, s) d\eta, \tag{6}$$

where $p_w(\eta, s)$ is the Wigner surmise for the Gaussian orthogonal ensemble with the mean spacing depending on the parameter,

$$p_w(\eta, s) = \eta s \exp(-\frac{1}{2}\eta s^2). \tag{7}$$

For a χ^2 distribution of the superstatistical parameter η the resulting NNS distribution is given by

$$p_{\chi^2}(\nu, s) = \frac{\eta_0 s}{(1 + \eta_0 s^2/\nu)^{1+\nu/2}}. \tag{8}$$

The parameter η_0 is fixed by requiring that the mean level spacing $\langle s \rangle$ equals unity, yielding

$$\eta_0 = \frac{\pi\nu}{4}[\Gamma(\frac{\nu - 1}{2})/\Gamma(\frac{\nu}{2})]^2. \tag{9}$$

For an inverse χ^2 distribution, one obtains

$$p_{inv\chi^2}(\nu, s) = \frac{2\eta_0 s}{\Gamma(\frac{\nu}{2})}(\sqrt{\eta_0 \nu s/2})^{\nu/2} K_{\nu/2}(\sqrt{\eta_0 \nu s}), \tag{10}$$

where $K_m(x)$ is a modified Bessel function [23] and η_0 again is determined by the requirement that the mean level spacing $\langle s \rangle$ equals unity. Finally, if the parameter has a log- normal distribution (5), then the NNS distribution

$$p_{LogNorm}(\nu; s) = \frac{s}{\sqrt{2\pi}\nu} \int_0^\infty \exp[-\frac{\eta s^2}{2} - \frac{\ln^2(\frac{2}{\pi}\eta e^{-\nu^2/4})}{2\nu^2}] d\eta \tag{11}$$

can only be evaluated numerically.

A justification for the use of the above-mentioned superstatistical generalization of RMT in the study of mixed systems, is given in [24]. It is based on the representation of their energy spectra in the form of discrete time series in which the level order plays the role of time. Reference [24] considers two billiards with mushroom-shaped boundaries as representatives of systems with mixed regular–chaotic dynamics and three with the shape of Limaçon billiards, one of them of chaotic and two of mixed dynamics. The time series analysis in Ref. [24] allows to derive a parameter distribution $f(\eta)$. The obtained distribution agrees better with the inverse χ^2 distribution given by Eq. (4). The inverse χ^2 distribution of η follows when the quantity η is the sum of ν inverse-squared Gaussian random variables. In the application to RMT, the parameter η is proportional to the sum of the inverse variances of the matrix elements of $\mathbf{H}$ [24]. Hence, ν refers to the number of (largely) contributing matrix elements. In the limit of $\nu \to \infty$ where all the matrix-elements contribute to the distribution, Eq. (4) yields a delta function for $p_{inv\chi^2}(\nu, s)$ turning superstatistical matrix-element distribution (2) into that of the conventional RMT, Eq. (1). If we take this assumption literarily, we must restrict ν to take positive integer values. As the transition from integrability to chaos is known to proceed continuously, we have to relax this condition and allow ν to take any real value grater than 1. Using the asymptotic expression of the modified Bessel function [23], we easily find the Wigner surmise when $\nu \to \infty$, as required. The other limit of $\nu \to 1$ yields the semi-Poisson distribution

$$p_{\mathrm{SS}}(1, s) = 4se^{-2s}, \tag{12}$$

which is known to provide a satisfactory description for the spectra of pseudointegrable systems such as planar polygonal billiards, when all their angles are rational with π [25]. The motion of the corresponding classical systems in phase space is not restricted to a torus like for integrable systems, but to a surface with a more complicated topology [26]. We therefore conclude that the assumption that the inverse square of the variance of matrix elements as an inverse χ^2 variable allows superstatistical RMT to model the transition out of chaos (corresponding to $\nu \gg 1$) until the system reaches the state of quasi-integrability.

It is interesting to note that, if one allows ν to take lower values, one finds that the

distribution (10) tends to the Poisson distribution as $\nu \to -1$;

$$p_{\mathrm{SS}}(-1, s) = e^{-s}. \tag{13}$$

We there conclude that formula (10) can provide a successful model for describing the stochastic transition all the way from integrability to chaos passing by the stage of quasi-integrability. This has been clearly demonstrated in Ref. [24]. We have no physical explanation for this success. We regard $p_{\mathrm{SS}}(\nu, s)$ in the range of $-1 \leq \nu \leq 1$ as a clever parametrization of NNS distribution of nearly integrable systems undergoing a transition from a Poisson to semi-Poisson statistics.

DATA ANALYSIS

Our purpose here is to show that the superstatistical RMT is suitable for the analysis of chaotic systems undergoing a process leading to the breaking of a discrete symmetry. In our view, symmetry breaking has something in common with the transition from integrability to chaos. The presence of a symmetry favors the particular bases in which the eigenvectors of the symmetry are components of the eigenstates of the Hamiltonian (e.g., the isospin wavefunctions in the nuclear physics problem). In this representation, the Hamiltonian matrix is block diagonal and its spectrum consists of independent sequence of eigenvalues. As symmetry breaking interactions increase, the eigenstates involved in the different blocks mix filling the "empty" places in the Hamiltonian matrix. At a certain stage of the symmetry breaking transition, the bases in which the eigenvectors of the symmetry are components of the eigenstates of the Hamiltonian loose their special status. The joint matrix-element distribution becomes base independent. At this point, we expect superstatistical RMT to become suitable for describing the symmetry breaking process with the superstatistical parameter ν measuring the number of effective matrix elements responsible for symmetry breaking.

In the quartz crystal experiment [1], the authors noted that the block with conserved flip symmetry has much in common with a scalar pseudointegrable system [25, 26]. The initial state of the transition can be described by an independent superposition of two independent semi-Poissonian sequences of equal densities. Applying the method given in Mehta's book

[11], we obtain for NNS distribution

$$p_{2\mathrm{PI}}(s) = \frac{1}{2} e^{-2s} \left(1 + 4s + 2s^2\right). \tag{14}$$

We have found out that the least square difference between $p_{2\mathrm{PI}}(s)$ and the superstatistical distribution (10) in the spacing interval $0 < s < 3.0$,

$$\int_0^{3.0} \left| p_{2\mathrm{PI}}(s) - p_{inv\chi^2}(\nu, s) \right|^2 \tag{15}$$

has a minimum at $\nu = -0.210$. In the following, we show that NNS distributions of resonances in the quartz crystal experiment [1], the authors noted that the initial state of the transition can be described by the distribution (10) allover the symmetry breaking transition by allowing ν to vary in the range of $-0.2 \leq \nu \leq \infty$.

In figure 1, we compare the experimental results of the acoustic resonances measured by Ellegaard et al. [1] with and the superstatistical NNS distributions corresponding to the χ^2, the inverse-χ^2, an the log-normal distributions of the superstatistical parameter. We also show in Fig. 1 the results of a previous analysis of the same data (figure 5 in Ref. [8]) with a random matrix model in which assumes that the spectra is composed of three independent components, two pseudo-integrable sequences for the conserved symmetry and one GOE sequence for the broken symmetry. The best-fit values of the parameters are given in Table 1. We quantify the quality of the fits by their absolute average deviation, $\Delta = \frac{1}{N_L} \sum \left| P_{Exp} - P_{Cal.} \right|$, where P_{Exp}, P_{Cal},and N_L are the experimental, the calculated NNS distributions and the number of measures values. The results suggests the validity of the superstatistical distribution, even in the initial stages of the breaking of the symmetry. The table clearly shown that the NNS distribution obtained from the inverse-χ^2 distribution (4) agrees with experiment data better than the other distributions even in the initial stages of the symmetry breaking process.

r	El-Hady et al	χ^2 distribution		inverse χ^2 distribution		Log normal distribution	
(mm)	Δ	ν_c	Δ	ν_i	Δ	ν_L	Δ
0	0.034	3.387	0.063	-0.112	0.035	1.100	0.053
0.5	0.037	2.887	0.061	-0.025	0.032	1.176	0.047
0.8	0.045	3.121	0.046	0.405	0.026	1.064	0.033
1.1	0.048	4.607	0.029	1.832	0.021	0.779	0.025
1.4	0.056	6.67	0.023	4.385	0.024	0.600	0.023
1.7	0.036	10.528	0.020	7.528	0.019	0.469	0.019
10	0.029	12.183	0.026	9.376	0.026	0.432	0.026

Table 1: The best fit parameters and absolute average deviations Δ for different radii r of the removed corner of the quartz block in the comparison of NNS distributions of acoustic resonances and the superstatistical RMT with a χ^2, inverse-χ^2 and log normal parameter distributions. The results of analysis of the three-level-sequence model by El-Hady et al. [8] are also shown

SUMMARY AND CONCLUSION

We have described the symmetry breaking of the acoustic resonance in a quartz blocks, using a superstatistical model that has been successfully applied to describe systems with mixed regular-chaotic dynamics within the framework of RMT. Superstatistical RMT is a base independent approach and is suitable to model symmetry breaking only when the symmetry representations become mixed enough. Superstatistics arises by allowing the mean density of states to fluctuate according to given distribution. We examined three possible parameter distributions, namely the χ^2, the inverse-χ^2 and the log normal distributions. Our analysis shows that the inverse-χ^2 distribution agrees with experimental spectra of acoustic resonances better than the other two, in the same way as in a previous analysis of stochastic transition of mixed microwave billiards. The superstatistical parameter ν that characterizes the fluctuation of the mean level density will hear measure the degree of symmetry breaking. We also show that NNS distributions with an inverse-χ^2 superstatistics provide a reasonable description of experiment data not only when the system approaches the state of chaos, but also in the initial stage of the symmetry breaking transition when base invariance is not expected.

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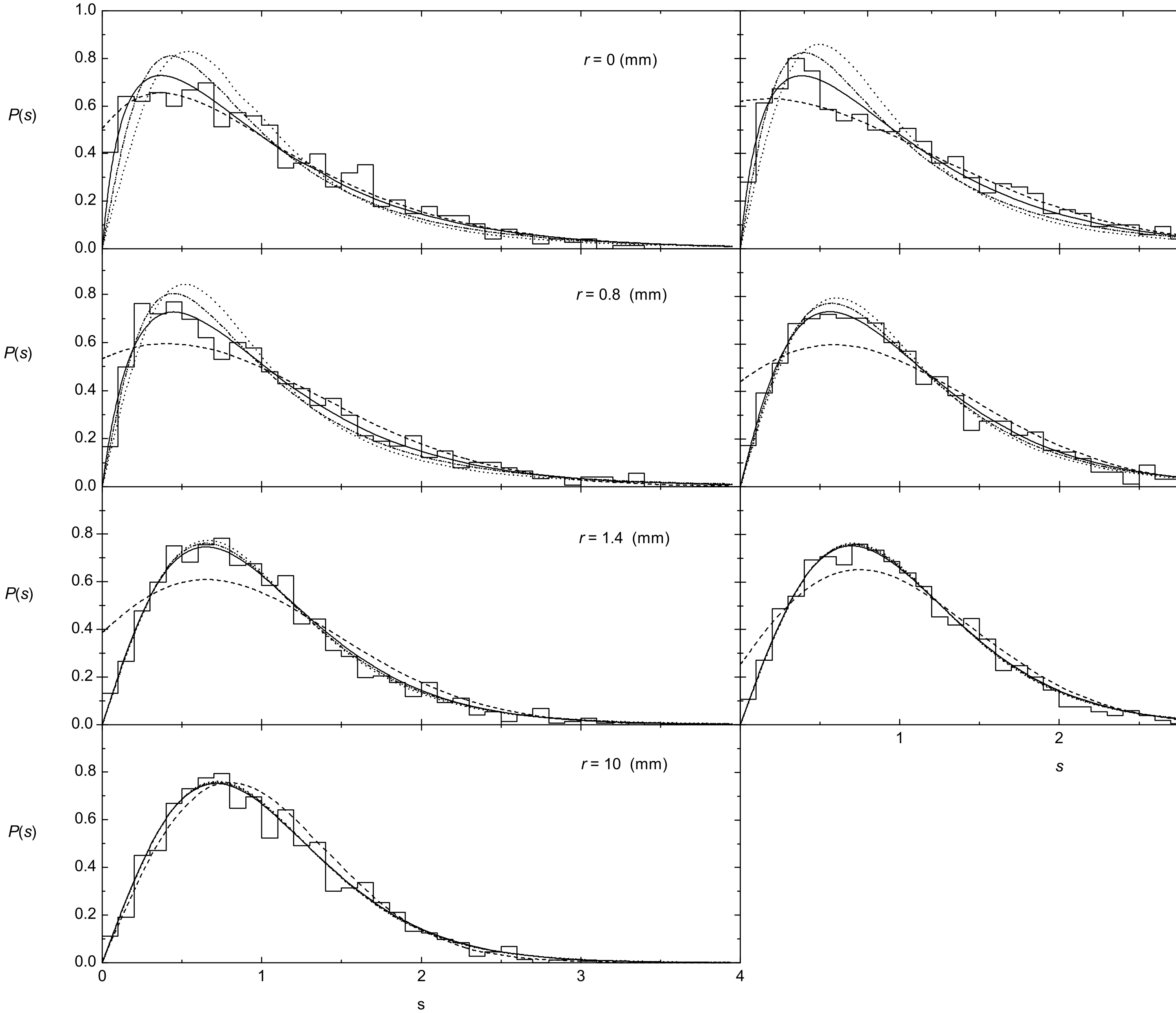
Figure Caption

Figure 1. NNS distributions for different radii r of the octant removed from the quartz blocks. The experimental data reported in [1] are shown by histgrams. The solid lines are the superstatistical results calculted with an inverse-χ^2 parameter distribution. Results for the χ^2 and log normal parameter distributions are shown by dotted and dashed dotted dotted crves, respectively. The dashed curves are calculated with the three-level-sequence model by El-Hady et al. [8].



r = 0 (mm)
r = 0.8 (mm)
r = 1.4 (mm)
r = 10 (mm)
P(s)
s